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STABILITY OF BUOYANCY-DRIVEN CONVECTION IN A TILTED SLOT

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Abstract—The stability of buoyancy-driven convection in a slot, slightly tilted with respect to the horizontal, is investigated analytically on the basis of linear theory. For mathematical simplicity, the boundaries are assumed free and isothermal. It is shown that the Rayleigh number and the wave number at the critical point have the same values as for an exactly horizontal slot, however, the predicted motion, rather than being indeterminate, is one of longitudinal rolls with their axes aligned in the direction of the mean flow. This is in contrast to the analogous problem of convection in a vertical slot in which the secondary flow pattern is known to consist of transverse rolls, i.e. rolls with their axes normal to the mean motion.

NOMENCLATURE

- c, amplification factor of the disturbance [see equation (4)];
- c_{p_0}, heat capacity, evaluated at the temperature T_0;
- d, depth of fluid layer;
- k_0, thermal conductivity of fluid, evaluated at the temperature T_0;
- P', pressure associated with the basic, i.e. the undisturbed, flow;
- P, $P'/\rho_0 U_c^2$;
- T_1, T_2, the temperatures at, respectively, the lower and upper plane;
- T_0, the arithmetic mean temperature $(T_1 + T_2)/2$;
- x, y, z, dimensionless Cartesian coordinates;
- u, v, w, dimensionless velocity components in the x, y, z direction, respectively;
- U_c, characteristic velocity $\equiv k_0/\rho_0 c_{p_0} d$;
- $U(y)$, the basic, i.e. unperturbed, velocity profile (dimensionless);
- $V(y)$, defined by equation (4);
- R, Rayleigh number

$$\frac{\rho c_p g \bar{\alpha}(T_1 - T_0)}{\nu k} d^3$$

(all physical properties evaluated at temperature T_0; g = gravitational acceleration; $\bar{\alpha}$ = coefficient of volumetric expansion).

Greek letters

- α, β, wave numbers in the x and z direction, respectively [see equation (4)];
- θ, dimensionless temperature $(T - T_0)/(T_1 - T_0)$;
- $\theta(\eta)$, defined by equation (4);
- ν, kinematic viscosity;
- ρ, fluid density;
- σ, Prandtl number $c_p\rho\nu/k$ (physical properties evaluated at T_0);
- φ, angle of inclination of the slot with respect to the horizontal;
- ξ, ζ, defined by equation (17).

Superscripts

- $'$, primes denote dimensional quantities;
- $\hat{}$, denotes perturbation quantities.

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1. INTRODUCTION

As a result of intensive study, considerable progress has been achieved over the past fifty years or so in understanding the subject of buoyancy-driven convection in fluid layers heated from below. For example, using a classical linear stability analysis, it is possible to predict the conditions for the onset of this convective motion, as well as the characteristic scale of the resulting flow pattern [1]. However, it is also well-known that, in a layer of unbounded horizontal extent, the flows that are possible according to linear theory form an infinitely degenerate set. Consequently, to obtain theoretically the actual flow structure that is realized physically in such a system, it is necessary to take the non-linear effects into account. Thus, it has been shown that convection near the critical point will consist of rolls if the fluid properties are temperature independent according to the Boussinesq approximation [2], or of hexagonal cells if these properties vary substantially across the fluid film [3, 4]. These theoretical predictions are in good agreement with experimental observations [5–7].

In such cases, the inability of linear theory to predict the flow pattern near the critical point results from the absence of any preferred direction along the two horizontal planes that confine an otherwise unbounded fluid layer. It might be expected, therefore, that a linear analysis would lead to a unique flow structure, or at least to a degeneracy of lower order, for systems in which such a complete isotropy did not exist. Indeed, as shown recently by Davis [8] and by Segel [9], when the fluid is contained in a rectangular box, finite rolls with their axes parallel to the shorter side are predicted on the basis solely of a linear treatment. This is consistent with Koschmieder's observations [6] who found that, near the critical point, the cell pattern that emerges is strongly influenced by the geometry of the lateral boundaries.

In this note we shall consider another example on this subject in which a unique flow pattern results from linear theory. This is the problem of buoyancy-driven convection in a fluid layer bounded by two infinite parallel surfaces, tilted at a small angle, φ, with respect to the horizontal. Here, a basic flow sets in which becomes unstable whenever the temperature difference between the two planes (with the bottom plane kept warmer than the top) exceeds a certain critical value. The similarity between this and the usual case in which the planes are exactly horizontal is of course evident; in fact, both the method of solution and some of the principal results of the linear stability analysis are almost identical. However, it will be seen that, although the critical wave number will remain unaffected by tilting the planes a small amount, a preferred mode will emerge in the form of rolls having their axes along the direction of the mean motion. Hence, owing to the existence of this basic flow which imparts a definite structure to the undisturbed system, the degeneracy usually associated with convection problems of this type will be removed.

2. BASIC EQUATIONS AND SOLUTION OF THE LINEAR STABILITY PROBLEM

The coordinate system is chosen as shown in Fig. 1. Here, for the sake of mathematical simplicity, the two boundary surfaces $y' = 0$, d are taken to be free and maintained at constant temperatures T_1 and T_2, respectively. It is convenient to introduce the non-dimensional quantities

$$(x, y, z) = \left(\frac{x'}{d}, \frac{y'}{d}, \frac{z'}{d}\right),$$

$$(u, v, w) = \left(\frac{u'}{U_c}, \frac{v'}{U_c}, \frac{w'}{U_c}\right), \qquad U_c = \frac{k_0}{\rho_0 c_{p_0} d}$$

$$P = \frac{P'}{\rho_0 U_c^2}, \qquad t = \frac{t' U_c}{d}, \qquad \theta = \frac{T - T_0}{T_1 - T_0},$$

$$T_0 = \tfrac{1}{2}(T_1 + T_2),$$

in which a prime refers to a dimensional variable and a subscript 0 to a physical property evaluated at the temperature T_0. Throughout this analysis the familiar Boussinesq approximation [1] will be invoked, according to which the physical properties are assumed to be temperature independent except for the density appearing in the buoyancy term.

Letting

$$\theta = 1 - 2y, \quad u = U(y), \quad v = w = 0,$$

it can easily be shown that the basic solution of the appropriate governing equations reduces to

$$u = U(y) = R \sin \varphi \left\{ \frac{y^3}{3} - \frac{y^2}{2} + \frac{1}{12} \right\}$$

$$v = w = 0, \qquad \theta = 1 - 2y \tag{1}$$

$$P = P_0 + \sigma R \cos \varphi (y - y^2) = P(y),$$

where P_0 is a constant, R is the Rayleigh number in terms of $T_1 - T_0$, and σ the Prandtl number. This solution indicates that no matter how small the inclined angle φ, a shear-like flow in the x-direction $[u = U(y)]$ will always be established, and that even in the presence of such a motion, the transport of heat from the lower to the upper plane will be due to conduction alone provided no lateral boundaries exist.

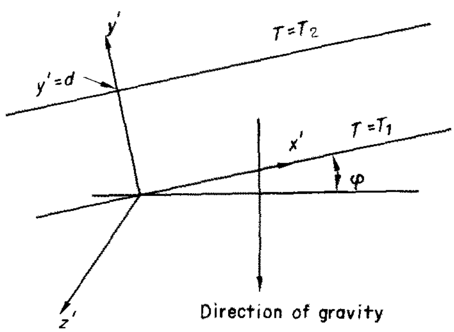

FIG. 1. The coordinate system.

Following the usual approach of linear stability theory, the following perturbation quantities

$$u = U(y) + \hat{u}(x, y, z), \qquad v = \hat{v}(x, y, z,), \qquad w = \hat{w}(x, y, z)$$

$$\theta = 1 - 2y + \hat{\theta}(x, y, z), \qquad P = P(y) + \hat{p}(x, y, z)$$

are next introduced into the basic equations. Neglecting the non-linear terms and dropping the carets yields

$$\frac{\partial u}{\partial t} + U(y)\frac{\partial u}{\partial x} + vDU = \sigma R \theta \sin \varphi - \frac{\partial p}{\partial x} + \sigma \nabla^2 u$$

$$\frac{\partial v}{\partial t} + U(y)\frac{\partial v}{\partial x} = \sigma R\, \theta \cos \varphi - \frac{\partial p}{\partial y} + \sigma \nabla^2 v$$

$$\frac{\partial w}{\partial t} + U(y)\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + \sigma \nabla^2 w \tag{2}$$

$$\frac{\partial \theta}{\partial t} + U(y)\frac{\partial \theta}{\partial x} - 2v = \nabla^2 \theta$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

where

$$D = \frac{\mathrm{d}}{\mathrm{d}y}, \qquad \text{and} \qquad \leftarrow^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Cross differentiating to eliminate the pressure term and further differentiating in x and z to allow elimination of u and w, gives

$$6\nabla^4 v - \frac{\partial}{\partial t}\nabla^2 v - U(y)\frac{\partial}{\partial x}\nabla^2 v + \frac{\partial v}{\partial x}D^2 U = \sigma R \sin \varphi \frac{\partial^2 \theta}{\partial x \partial y} - \sigma R \cos \varphi \nabla_1^2 \theta \tag{3}$$

$$\left\{ \nabla^2 - \frac{\partial}{\partial t} - U(y)\frac{\partial}{\partial x} \right\} \theta + 2v = 0,$$

where

$$\nabla_1^2 = \nabla^2 - D^2.$$

Equation (3) can be simplified by assuming solutions of the form

$$v(x, y, z, t) = V(y) \exp\{\mathrm{i}(\alpha x + \beta z - ct)\}$$
$$\theta(x, y, z, t) = \theta(y) \exp\{\mathrm{i}(\alpha x + \beta z - ct)\} \tag{4}$$

of which the real parts represent the actual physical quantities. The wave numbers, α and β, are real and the growth rate, c, is generally complex. In terms of (4), equation (3) becomes

$$\left(\sigma(D^2-\alpha^2-\beta^2)^2 - i\alpha\left\{\left[U(y)-\frac{c}{\alpha}\right]\times[D^2-\alpha^2-\beta^2] - D^2U\right\}\right)V(y)$$

$$= \sigma R\sin\varphi\cdot i\alpha D\theta(y) + \sigma R(\alpha^2+\beta^2)\theta(y)\cdot\cos\varphi$$

$$[D^2-\alpha^2-\beta^2+ic-i\alpha U(y)]\,\theta(y) + 2V(y) = 0, \quad (5)$$

with boundary conditions

$$V(y) = D^2V(y) = \theta(y) = 0 \quad \text{at} \quad y = 0, 1. \quad (6)$$

Equation (5) is the familiar Orr–Sommerfeld equation coupled with the energy equation. Recently, using the Galerkin method, Vest and Arpaci [10] developed an approximate solution to this system for the case $\varphi = 90°$, and their results will be discussed later on. Here, since the present study is restricted to small inclined angles, φ, the above equation will be solved by a much simpler perturbation technique using $\sin\varphi$ as a small perturbation quantity. Thus, expanding the solutions of equation (5) into the form:

$$\{c,\alpha,\beta,R,V(y),\theta(y)\} = \{c_0,\alpha_0,\beta_0,R_0,V_0(y),\theta_0(y)\} + \{c_1,\ \alpha_1,\beta_1,R_1,V_1(y),\theta(y)\}\sin\varphi + \{c_2,\alpha_2,\beta_2,R_2,V_2(y),\theta_2(y)\}\sin^2\varphi + \dots \quad (7)$$

we obtain for the zeroth order system

$$[\sigma(D^2-\alpha_0^2-\beta_0^2)+ic_0]\,(D^2-\alpha_0^2-\beta_0^2)\,V_0(y) - \sigma R_0(\alpha_0^2+\beta_0^2)\,\theta_0(y) = 0$$

$$(D^2-\alpha_0^2-\beta_0^2+ic_0)\theta_0(y) + 2V_0(y) = 0.$$

Here, as is well known [1], the principle of the exchange of stabilities applies, hence ic_0 is real and the marginal state is characterized by $c_0 = 0$. The solution is readily available and is given by

$$V_0(y) = \sin\pi y, \qquad \theta_0(y) = \frac{4}{3\pi^2}\sin\pi y,$$

$$\alpha_0^2+\beta_0^2 = \frac{\pi^2}{2}, \qquad R_0 = \frac{27\pi^4}{8}. \quad (8)$$

It should be pointed out that R_0 equals $27\pi^4/8$ instead of $27\pi^4/4$ as given in the standard references, [1], because the Rayliegh number was defined above in terms of half the temperature difference between the two bounding surfaces, $y = 0, 1$.

Before solving the higher order equations, it is first necessary to solve the homogeneous adjoint problem. Applying a method similar to that used in analogous studies [2], it can be shown that the latter is given by

$$\sigma(D^2-\alpha_0^2-\beta_0^2)^2\,V^*(y) + 2\theta^*(y) = 0$$

$$(D^2-\alpha_0^2-\beta_0^2)\,\theta^*(y) - \sigma R_0(\alpha_0^2+\beta_0^2)\,V^*(y) = 0$$

and that the boundary conditions are the same as (6). Hence

$$V^*(y) = V_0(y) = \sin\pi y,$$

$$\theta^*(y) = -\frac{27\pi^6}{32}\sigma\theta_0(y) = -\sigma\frac{9\pi^4}{8}\sin\pi y. \quad (9)$$

Substituting (7) into (5), we next obtain for the first order equations

$$\sigma(D^2-\alpha_0^2-\beta_0^2)^2\,V_1 - \sigma R_0(\alpha_0^2+\beta_0^2)\,\theta_1$$
$$= 4(\alpha_0\alpha_1+\beta_0\beta_1)\,\sigma(D^2-\alpha_0^2-\beta_0^2)\,V_0 - i\alpha_0R_0V_0D^2h + i\alpha_0R_0h(D^2-\alpha_0^2-\beta_0^2)\,V_0 - ic_1(D^2-\alpha_0^2-\beta_0^2)\,V_0 + i\alpha_0\sigma R_0D\theta_0 + \sigma R_1(\alpha_0^2+\beta_0^2)\,\theta_0 + 2\sigma R_0(\alpha_1\alpha_0+\beta_1\beta_0)\,\theta_0,$$

$$(D^2-\alpha_0^2-\beta_0^2)\,\theta_1 + 2V_1 = i\alpha_0R_0h\theta_0 - ic_1\theta_0 + 2(\alpha_1\alpha_0+\beta_1\beta_0)\,\theta_0 \quad (10)$$

where $h(y)$ is given by

$$h(y) \equiv \frac{U(y)}{R\sin\varphi} = \left(\frac{y^3}{3} - \frac{y^2}{2} + \frac{1}{12}\right).$$

Since the inhomogeneous part of equation (10) must be orthogonal to the homogeneous adjoint solution, the eigenvalue, R_1, can be computed as follows: Multiplying the first equation in (10) by V^* and the second by θ^*, summing and then integrating from $y = 0$ to $y = 1$, yields

$$R_1 = -\frac{9\pi^2}{4}\mathrm{i}c_1\left(1 + \frac{1}{\sigma}\right).$$

Since R_1 is real, c_1 must be imaginary. Thus, to this order, there is no oscillatory motion and at the neutral state, c_1 and hence R_1 must equal zero.

In view of (8) and the fact that c_1 and R_1 are both equal to zero, equation (10) becomes

$$\sigma\left(D^2 - \frac{\pi^2}{2}\right)^2 V_1(y) - \frac{27\pi^6}{16}\sigma\theta_1(y) = \mathrm{i}\alpha_0\frac{9\pi^3}{2}\sigma\cos\pi y - \mathrm{i}\alpha_0\left\{\frac{81\pi^6}{16}\left(\frac{y^3}{3} - \frac{y^2}{2} + \frac{4y}{3\pi^2} + \frac{1}{12} - \frac{2}{3\pi^2}\right)\right\}\sin\pi y + 3\pi^2\sigma(\alpha_1\alpha_0 + \beta_1\beta_0)\sin\pi y, \quad (11)$$

$$\left(D^2 - \frac{\pi^2}{2}\right)\theta_1(y) + 2V_1(y) = \mathrm{i}\alpha_0\frac{9\pi^2}{2}\left(\frac{y^3}{3} - \frac{y^2}{2} + \frac{1}{12}\right)\sin\pi y + (\alpha_1\alpha_0 + \beta_1\beta_0)\cdot\frac{8}{3\pi^2}\sin\pi y. \quad (12)$$

Further elimination of θ_1 yields

$$\left\{\left(D^2 - \frac{\pi^2}{2}\right)^3 + \frac{27\pi^6}{8}\right\}\bar{V}_1(y) = \frac{1}{\sigma}\left\{\frac{81\pi^7}{8}\left[y - y^2 - \frac{2}{3\pi^2}(2 + \sigma)\right]\cos\pi y + \frac{81\pi^8}{128}(1 + \sigma)(4y^3 - 6y^2 + 1)\sin\pi y\right\} \quad (13)$$

where

$$\bar{V}_1(y) = \frac{1}{\mathrm{i}\alpha_0}V_1(y),$$

with boundary conditions

$$\bar{V}_1 = D^2\bar{V}_1 = 0 \quad \text{at} \quad y = 0, 1$$

$$D^4\bar{V}_1 = \frac{9\pi^3}{2} \quad \text{at} \quad y = 0,$$

$$D^4\bar{V}_1 = -\frac{9\pi^3}{2} \quad \text{at} \quad y = 1.$$

Equation (12) can be simplified by substituting

$$\theta_1(y) = \mathrm{i}\alpha_0\bar{\theta}_1(y) + 2(\alpha_1\alpha_0 + \beta_1\beta_0)\tilde{\theta}_1(y).$$

Then

$$\left(D^2 - \frac{\pi^2}{2}\right)\bar{\theta}_1(y) = -2\bar{V}_1 + \frac{9\pi^2}{2}\left(\frac{y^3}{3} - \frac{y^2}{2} + \frac{1}{12}\right)\sin\pi y \quad (14)$$

and

$$\left(D^2 - \frac{\pi^2}{2}\right)\tilde{\theta}_1(y) = \frac{4}{3\pi^2}\sin\pi y. \quad (15)$$

Clearly,

$$\tilde{\theta}_1(y) = -\frac{8}{9\pi^4}\sin\pi y.$$

As for $\bar{V}_1(y)$ and $\bar{\theta}_1(y)$, these had to be obtained via a numerical solution of equations (13) and (14) and are shown in Figs. 2 and 3. As required by their governing equations and the associated boundary conditions, both functions are antisymmetric with respect to the mid-point $y = 0{\cdot}5$. Also, it is apparent from Fig. 2 that, for Prandtl numbers higher than 1·0, the magnitude of $2\bar{V}_1$ is much less than that of the other term in the right hand side of equation (14), hence one would expect $\bar{\theta}_1(y)$ to be very insensitive to the Prandtl number. This was borne out by the numerical solutions to equation (14) which, for $\sigma \geqslant 1$, could be represented to within a few per cent by means of $\bar{\theta}_1(y) \cong -0{\cdot}04\sin 2\pi y$.

Since the eigenvalue R_1 is zero at the neutral state, it is necessary to compute R_2. This can be achieved from the solvability condition of the second order equation. The latter is given by

$$\sigma\left(D^2 - \frac{\pi^2}{2}\right)^2 V_2 - \sigma\frac{27\pi^6}{16}\theta_2 = \alpha_0^2\left\{R_0\bar{V}_1 D^2 h - R_0 h\left(D^2 - \frac{\pi^2}{2}\right)\bar{V}_1 - R_0\sigma D\bar{\theta}_1\right\}$$
$$+ \mathrm{i}\alpha_0\xi\left\{\sigma R_0 D\bar{\theta}_1 - R_0 h V_0 + 2\sigma\left(D^2 - \frac{\pi^2}{2}\right)\bar{V}_1 + \sigma R_0\bar{\theta}_1\right\} + \mathrm{i}\frac{3\pi^2}{2}c_2\sin\pi y + \xi^2(\sigma R_0\bar{\theta}_1 - \sigma V_0)$$
$$- \frac{\pi^2}{4}\sigma R_0\theta_0 + \frac{\pi^2}{2}\sigma R_2\theta_0 + \mathrm{i}\alpha_1\left\{R_0 h\left(D^2 - \frac{\pi^2}{2}\right)V_0 - R_0 V_0 D^2 h + \sigma R_0 D\theta_0\right\}$$
$$+ \zeta\left\{2\sigma\left(D^2 - \frac{\pi^2}{2}\right)V_0 + \sigma R_0\theta_0\right\},$$
$$\left(D^2 - \frac{\pi^2}{2}\right)\theta_2 + 2V_2 = -\mathrm{i}c_2\theta_0 + \xi^2\bar{\theta}_1 + \zeta\theta_0 + \mathrm{i}\alpha_0 R_0 h\bar{\theta}_1\xi + \mathrm{i}\alpha_0\xi_1 - \alpha_0^2 R_0 h\bar{\theta}_1 + \mathrm{i}\alpha_1 h R_0\theta_0 \quad (16)$$

where

$$\xi = 2\alpha_1\alpha_0 + 2\beta_1\beta_0, \qquad \zeta = 2\alpha_2\alpha_0 + \alpha_1^2 + 2\beta_2\beta_0 + \beta_1^2. \quad (17)$$

Again, multiplying the first equation in (16) by V^* and the second by θ^*, summing and integrating, yields

$$R_2 = \tfrac{1}{2}R_0 - \alpha_0^2 R_0 k_1(\sigma) - 3\mathrm{i}\alpha_0\xi k_2(\sigma) + \tfrac{9}{2}\xi^2 - \mathrm{i}c_2\frac{9\pi^2}{4}\left(1 + \frac{1}{\sigma}\right) \quad (18)$$

where

$$k_1(\sigma) = \frac{3}{\sigma}\int_0^1\left\{\left[\bar{V}_1 D^2 h - h\left(D^2 - \frac{\pi^2}{2}\right)\bar{V}_1 - \sigma D\bar{\theta}_1\right]V^* - h\bar{\theta}_1\theta^*\right\}\mathrm{d}y$$

and

$$k_2(\sigma) = \frac{1}{\sigma}\int_0^1\left\{V^*\left[2\sigma\left(D^2 - \frac{\pi^2}{2}\right)\bar{V}_1 + \sigma R_0\bar{\theta}_1\right] + \bar{\theta}_1\theta^*\right\}\mathrm{d}y$$

Since both $\bar{V}_1(y)$ and $\bar{\theta}_1(y)$ are anti-symmetric with respect to $y = 0{\cdot}5$, $k_2(\sigma)$ must vanish. As for $k_1(\sigma)$, this was evaluated numerically at different Prandtl numbers and is seen tabulated in Table 1. Clearly, $k_1(\sigma)$ is everywhere negative.

At the neutral state, c_2 must be real. Since R_2 is real, the real and imaginary parts of equation (18) reduce, respectively, to

$$R_2 = \tfrac{1}{2}R_0 - \alpha_0^2 R_0 k_1(\sigma) + \tfrac{9}{2}\xi^2 \quad (19)$$

and

$$c_2 = -\frac{4}{3\pi^2}\frac{\sigma}{1+\sigma}\alpha_0\xi k_2(\sigma) = 0.$$

To this order then, no oscillatory motions are possible at the neutral state. This is to be expected, since the absence of a preferred direction for wave travel would suggest a stationary instability.

Owing to the fact that $k_1(\sigma)$ is negative, it is obvious that R_2 has a minimum value of $\frac{1}{2}R_0$ when both α_0 and ξ vanish. With α_0 and ξ both zero, it is apparent from (8) and (17) that β_0 has a value of $\pi/\sqrt{2}$ and that β_1 is zero. Furthermore, both $V_1(y)$ and $\theta_1(y)$ are also zero. Thus, summarizing the results obtained so far, we have that, at the neutral state,

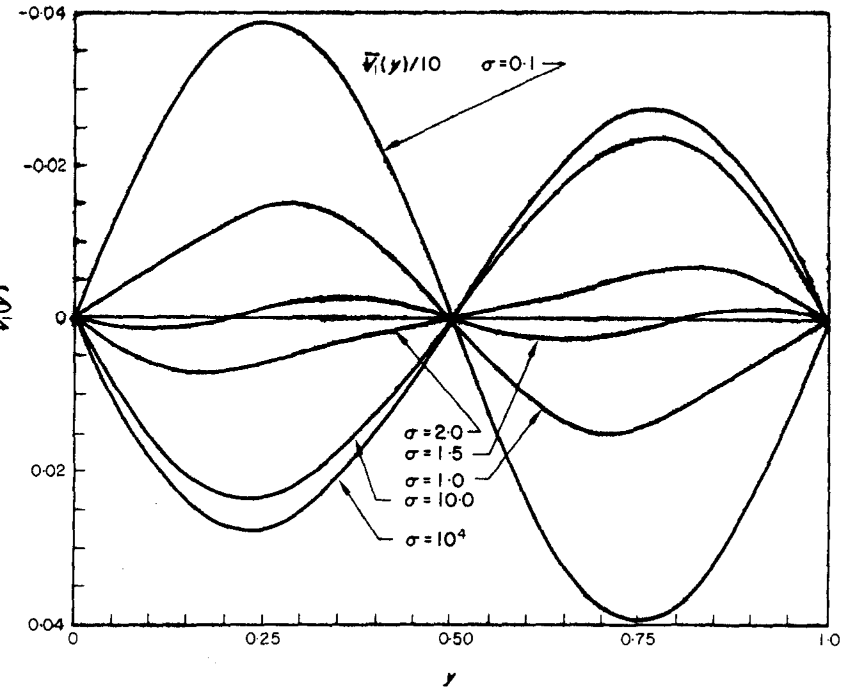

FIG. 2. The function $\bar{V}_1(y, \sigma)$.

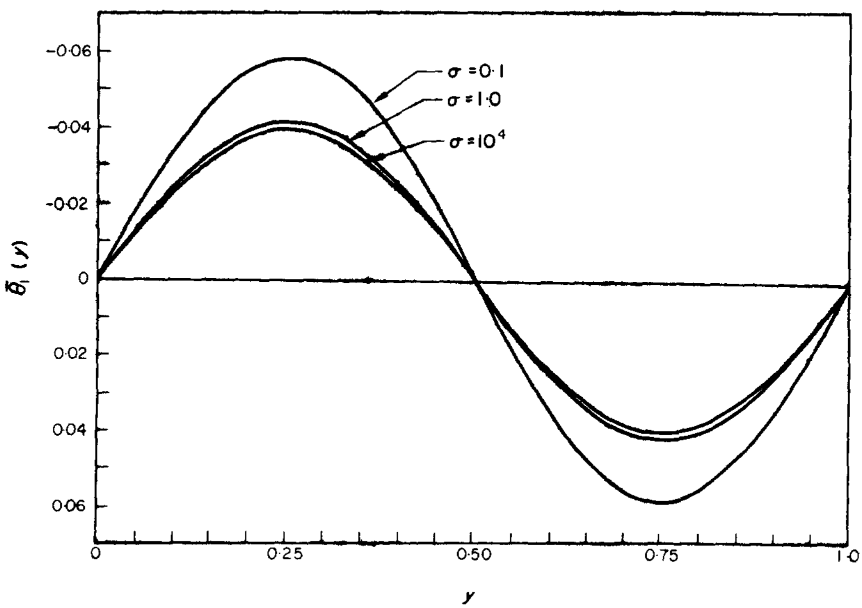

FIG. 3. The function $\bar{\theta}_1(y, \sigma)$.

$$
\begin{aligned}
\alpha &= \alpha_1 \sin\varphi + O(\sin^2\varphi) \\
\beta &= \frac{\pi}{\sqrt{2}} + O(\sin^2\varphi),\; c = O(\sin^3\varphi) \\
R &= \frac{27\pi^4}{8}(1 + \tfrac{1}{2}\sin^2\varphi) + O(\sin^3\varphi) \qquad (20) \\
V(y) &= \sin\pi y + O(\sin^2\varphi) \\
\theta(y) &= \frac{4}{3\pi^2}\sin\pi y + O(\sin^2\varphi).
\end{aligned}
$$

It should be pointed out that if the Rayleigh number were defined in terms of $g\cos\varphi$ rather than g, then, to $O(\sin^2\varphi)$, the critical Rayleigh number given above would simply reduce to R_0. In addition, it is evident from (4) that this neutral state corresponds to a longitudinal roll solution, i.e. to rolls parallel to the x-axis.

3. DISCUSSION

According to the results presented above, the Rayleigh number at the neutral state will have a minimum value $R_0 \cos\varphi$ for a steady longitudinal roll disturbance with its axis parallel to the direction of the basic flow and with wave number β equal to that of the corresponding solution for an exactly horizontal layer. For other disturbances, i.e. for $\alpha_0 \neq 0$, the critical Rayleigh number would be given by equation (19) and oscillatory motions would again be excluded. Thus, the linear stability theory, when applied to the system depicted in Fig. 1, leads to the very interesting prediction that, if the Rayleigh number is slowly increased past the critical point, the ensuing convective motion for small values of φ will consist of steady parallel rolls having a definite wave length and with their axes in the x-direction.

These theoretical conclusions are in a sense rather similar to those arrived at by Gallagher and Mercer [11] and by Deardorff [12] who examined the problem of gravitational instability between horizontal plates in the presence of a small shear flow, a problem which has many points in common with that considered here. These authors found that, as in the present case, the critical Rayleigh number is smallest, and in fact the same as that for pure convection without shear, for longitudinal roll disturbances having their axes aligned in the direction of the mean flow (i.e. $\alpha_0 = 0$). For the other disturbances ($\alpha_0 \neq 0$), which generally lead to an oscillatory instability, the critical Rayleigh number was found to increase sharply with increasing Prandtl number. This last result differs from that obtained here in two respects: first, it was shown that the neutral state in the present problem remains stationary for all disturbance wave numbers. In addition, since the absolute value of $k_1(\sigma)$ approaches a constant as the Prandtl number is increased, the critical Rayleigh number as given by (19) will also increase to an asymptotic value independent of σ.

Undoubtedly, the problem bearing the closest resemblance to that being considered here is that of the stability of natural convection in a vertical slot. As mentioned earlier, this was studied by Vest and Arpaci [10] who found that, owing to the absence of the term in equations (3) proportional to cos φ, Squire's theorem could be extended to this problem. Consequently, the predicted flow pattern at the critical point, which was shown to be stationary and to be characterized by the Grashof rather than the Rayleigh number as is the case here, was that of transverse rolls, i.e. rolls having their axes

Table 1. Values of $k_1(\sigma)$

σ	0·1	1·0	2·0	10	10^4
$-k_1(\sigma)$	9·081	0·470	0·424	0·417	0·419

normal to the direction of the mean flow. This last result, confirmed experimentally by Vest and Arpaci, differs of course from that of the present analysis in which longitudinal rolls were predicted at the point of instability. Thus, it would appear that, as φ is increased from 0 to 90°, a transition, perhaps even a sharp one, should take place from stationary longitudinal to stationary transverse rolls. It would be of interest to try and observe this transition experimentally.

The predicted flow pattern at the neutral state also seems to depend on whether the instability is primarily of hydrodynamic or of thermal (convective) origin. For example, thermal instability occurs when the layer is nearly horizontal and is heated from below, as in the present case. In contrast, when the layer is vertical or is positioned at such an angle that it corresponds to heating from above, then the mechanism of instability is hydrodynamic, i.e. it refers to the instability of two opposing convective streams. Within the transition range of the angle of inclination both mechanisms are active and lead to a rather complicated dependence of the critical Rayleigh number of φ and σ which was recently determined by Birikh *et al.* [13] for the special case of transverse rolls.

Before closing, it is perhaps worth remarking that, although the present analysis has dealt for reasons of mathematical simplicity only with the case of free, isothermal boundaries along the two planes, past experience would indicate that the principal conclusions of this study would not have been affected by the use of more realistic boundary conditions. However, to show this rigorously would have required considerable effort.

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STABILITÉ DE LA CONVECTION DUE À LA BRAVITÉ DANS UNE FENTE INCLINÉE

Résumé—La stabilité de la convection due à la gravité dans une fente, légèrement inclinée par rapport à l'horizontale, est étudiée analytiquement sur la base de la théorie linéaire. Pour la simplicité mathématique, on suppose que les frontières sont libres et isothermes. On montre que le nombre de Rayleigh et le nombre d'onde au point critique ont les mêmes valeurs que pour une fente exactement horizontale; cependant, le mouvement prévu, au lieu d'être indéterminé, consiste en rouleaux longitudinaux avec des axes alignés dans la direction de l'écoulement moyen. Ceci est en opposition avec le problème analogue de la convection dans une fente verticale dans lequel on sait que l'écoulement secondaire consiste en rouleaux transversaux, c'est-à-dire, en rouleaux avec des axes normaux à l'écoulement moyen.

STABILITÄT DER NATÜRLICHEN KONVEKTION IN EINEM GENEIGTEN SPALT

Zusammenfassung—Die Stabilität der natürlichen Konvektion in einem Spalt der leicht gegen die Horizontale geneigt ist, wird analytisch auf Grund der Lineartheorie untersucht. Zur mathematischen Vereinfachung sind die Berandungen als frei und isotherm angenommen. Es wird gezeigt, dass die Rayleigh-Zahl und die Wellenzahl am kritischen Punkt die gleichen Werte geben, wie beim genau horizontalen Spalt, doch ist die vorhergesagte Bewegung nicht unbestimmt, sondern eher eine Bewegung von Längsrollen deren Achsen in Richtung der mittleren Strömung ausgerichtet sind. Dies steht im Gegensatz zum analogen Problem der Konvektion in senkrechten Spalten in welchen das Sekundärströmungsmuster aus Querrollen besteht, d.h. Rollen deren Achsen normal zur mittleren Bewegungsrichtung stehen.

УСТОЙЧИВОСТЬ ТЕЧЕНИЯ ПРИ НАЛИЧИИ ЕСТЕСТВЕННОЙ КОНВЕКЦИИ В НАКЛОННОЙ ЩЕЛИ

Аннотация—На основе линейной теории аналитически исследуется устойчивость течения при наличии естественной конвекции в слабо наклонённой относительно горизонтали щели. Для простоты математических расчетов принято, что твердые границы являются бесконечными и изотермическими. Показано, что критерий Релея и волновое число в критической точке имеют те же значения, что и для щели, расположенной строго горизонтально, однако расчетное течение имеет вид продольных валов с осями, ориентированными в направлении основного движения. Этот результат противоположен результату, полученному при решении аналогичной задачи конвекции в вертикальной щели, когда вторичное течение состоит из поперечных валов, т.е. валов с осями, расположенными перпендикулярно основному движению.